Proving Theorems from Reflection

P.D. Welch
School of Mathematics,
University of Bristol,
Bristol, BS8 1TW, England

Revised 14 August 2018

.

Abstract

We review some fundamental questions concerning the real line of mathematical analysis, which, like the Continuum Hypothesis, are also independent of the axioms of set theory, but are of a less 'problematic' nature, as they can be solved by adopting the right axiomatic framework. We contend that any foundations for mathematics should be able to simply formulate such questions as well as to raise at least the theoretical hope for their resolution.

The usual procedure in set theory (as a foundation) is to add so-called strong axioms of infinity to the standard axioms of Zermelo-Fraenkel, but then the question of their justification becomes to some people vexing. We show how the adoption of a view of the universe of sets with classes, together with certain kinds of *Global Reflection Principles* resolves some of these issues.

1 Introduction

This essay falls into two distinct parts. We first look at some long-standing questions in mathematical analysis, from the Russian and French schools of the early 20'th century, and how they have, or have not, been answered since. Our purpose here is two-fold: to step away from the eternal recurrence of Cantor's Continuum Problem in debates of this kind, which is a question in third order number theory, to give examples in second order number theory, or what logicians would also call plain 'analysis'. Our second purpose is to here make the case that the questions considered are natural ones in the context of mathematical thought. Few mathematical analysts ever come across a problem where the continuum hypothesis, that $2^{\aleph_0} = \aleph_1$, is ever an important consideration, and they are aware of its independence from the other ZF axioms. Questions such as whether projections of co-analytic sets are Lebesgue measurable, for example, are much nearer their domains of interest. If a mathematician wants to know whether such a set A, say is Lebesgue measurable, or has meagre symmetric difference from an open set, we cannot wish this question away by talking about a 'multiverse', or the dependence of its truth on some model of set theory obtained by forcing, or on some variant foundational theory or other: they want to know the answer.

Our not so - hidden agenda then, is to make the point that any foundation of mathematics has to be able to both simply formulate these questions, since they are naturally occurring statements of mathematical significance, even to the extent of their being simply written, and moreover to give some succour at least to the possibility of their resolution.

The second part is rather different. One advance over the independence phenomena ushered in by Cohen, has been for set theorists to expand the axioms of Zermelo-Fraenkel set theory (ZF, or ZFC with the Axiom of Choice added) by so-called 'strong axioms of infinity' often phrased in terms of 'large cardinal' numbers (actually it is not their largeness, but the strong or exotic properties they bear, that yields their strength). The question of justification of the assumption of such axioms then looms large. (But perhaps it is only a larger worry for the foundationalist than for the mathematician: when Andrew Wiles was asked whether it would bother him if the unbounded class of Grothendieck universes (and hence a proper class of inaccessible cardinals), that *prima facie* had been invoked for his proof of Fermat's Last Theorem, turned out to be necessary for his argument, his reaction was a metaphorical shrug when not a literal one: not in the slightest. It was neither here nor there; in short he had a proof. The point of the story is that the mathematics was already convincing.)

We give a straightforward account of much of this that is familiar to set theorists, but perhaps not elsewhere, and in the second part (section 4) we deal with a recent proposal that notions of 'reflection' on the universe of sets instituted by early researchers such as Ackermann and Bernays, and warmly endorsed by Gödel, can be expanded in ways to demonstrate the existence of such large cardinals that solve the problems we give in the first part.

We should like to emphasise that our contribution in the first part is limited only to exposition and is indebted, amongst others, to [28], [29] and to any general history of descriptive set theory. The reader will find the descriptive set theory they need in Moschovakis [12].

We should like to warmly thank the referee who saved us from more than one embarrassing infelicity.

2 The task to hand

We look at some problems in the *projective hierarchy* of Luzin. However first we give Borel's hierarchy. In the following a 'Polish space' is a separable, complete metrisable space. This includes the common examples of the reals $\mathbb{R}$ with the usual Euclidean metric, Baire space $\mathbb{N}^{\mathbb{N}}$, and Cantor space $2^{\mathbb{N}}$ with metrics derived from the standard product topologies.

Definition 2.1 (Lebesgue (1905) Borel Sets) *Let T be a Polish space; let B_0 be the class of closed sets in T ;*
Let $B_\eta = \{\bigcup_{n\in\mathbb{N}} A_n \mid \neg A_n$ in some B_{η_n} for an $\eta_n < \eta\}$.
Let $\mathcal{B} = \bigcup_{\eta<\omega_1} B_\eta$.

Implicit in the definition above is that we perform complementation and union throughout all the countable ordinals, and the process finishes at stage ω_1 - the first uncountable cardinal: nothing further would result from continuing further. This analysis of a certain sequence of easily described sets into a hierarchy is a step in so-called 'descriptive set theory' that seeks to analyse the real line (or other nearby Polish space examples) in terms of a hierarchy of increasing complexity.

Definition 2.2 (Suslin (1917): Analytic Sets) *Let T be a Polish space; let $\mathcal{B}$ be the class of Borel sets* *in $T \times T$; let*

$$\mathcal{A} =_{df} \{A \mid \exists C \in \mathcal{B}(A = proj(C)\}$$

where $proj(C) = \{x \in T \mid \exists y \in T : C(x, y)\}$. Then $\mathcal{A}$ is called the class of analytic *sets.*

Theorem 2.3 (Suslin (1917)) *Borel = $\mathcal{A}$ & co-$\mathcal{A}$, that is the Borel sets in a space are precisely those analytic sets with analytic complement.*

So here we have written co-$\mathcal{A}$ for the class of sets whose complement is in $\mathcal{A}$ (and similarly will do so below "co-S" for other classes S). Descriptive set theorists would call the class of Borel sets the 'dual' class of $\mathcal{A}$. The study of the projective hierarchy was initiated by the discovery of Suslin (a student at the time) that Lebesgue had erred in assuming the projection of a Borel set was Borel. It was not. Indeed there was a hierarchy of sets to be investigated obtained by projection and complementation:

Definition 2.4 (Luzin (1925), Sierpiński (1925) The Projective Sets) *Let T be a Polish space.*

$$S_1 = \mathcal{A} \subseteq T^k \textit{(in any dimension)}; \; S_{n+1} = \{proj(D) \mid D \subseteq T^k \times T, D \in co\text{-}S_n\}; \quad PROJ = \bigcup_n S_n.$$

Lebesgue studied these and showed that they formed a proper hierarchy of increasing complexity as n increased. Sierpinsky later showed they were closed under countable unions and intersections. It is important to realise that these are the *definable* sets in analysis: the operations of projection and complementation in the above definition, correspond when written out even in informal notation to an existential quantification over the elements of T and to negation. With T equalling $\mathbb{R}$, this means that any definition of a set of reals the analyst writes down will fall inside the class $PROJ$.

The following intimates that the projective sets might be very *regular*: in this case that they can always be assigned a meaningful, length, area, volume...

Theorem 2.5 (Suslin (1917)) *Any $D \in \mathcal{A}$ is Lebesgue measurable.*

However there the matter lay stuck. Attempts to ascend the projective hierarchy and establish, for example the Lebesgue property conspicuously failed.

Problem 1 (Lebesgue Measurability) *Are the sets in $PROJ$ Lebesgue measurable?*

It seemed intractable:

> *(Luzin - 1925) "One does not know and one will never know whether the projective sets are Lebesgue measurable".*

The Baire and Perfect Subset properties

A set U is said to have the *Baire Property (BP)* if it has meagre symmetric difference with some open set. (In turn a set is *meagre* if it is the countable union of nowhere dense sets. In some sense it is 'negligible'.) It was known (Lusin and Sierpiński -1923 [7]) that analytic sets (and so also their complements) had the Baire property.

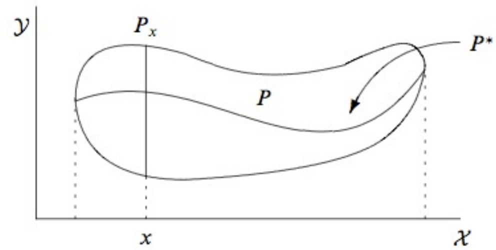
Problem 2 (Property of Baire) *Do sets in PROJ have the property of Baire (BP)?*

A perfect set is one which is closed but contains no isolated points. Since a perfect set has size the continuum, Cantor's continuum problem is settled for such sets.

Problem 3 (Perfect subset property (PSP)) *Does every uncountable set in PROJ contain a perfect set?*

It was known (due to Suslin) that every uncountable analytic set contained a perfect subset. (This may fail for co-analytic sets, for example in the Gödel constructible hierarchy.)

Uniformisation



A function $P^* \subseteq P \subseteq \mathcal{X} \times \mathcal{Y}$ *uniformizes* P if

$$\forall x[\exists y(x, y) \in P \to \exists\, y' \,(P^*(x) = y' \wedge (x, y') \in P)].$$

A function P^* is *projective* if its graph is.

Problem 4 (Uniformization Property (*Unif*)) *Does every set P in $T \times T$ in PROJ have a projective uniformizer? To abbreviate: $Unif(PROJ)$?*

For co-analytic sets a classical theorem yields that there is always a projective uniformizing function moreover one of the same complexity.

Theorem 2.6 (Novikov-Kondō 1937) *Every co-analytic subset of the plane has a co-analytic uniformizer.*

The above properties of the projective sets are called the *regularity properties*. For good measure we add one more.

The Banach-Tarski Property

Problem 5 (Banach -Tarski Problem) *Is there a paradoxical decomposition of the sphere in R^n into projective pieces?*

The original Banach-Tarski theorem states that it is possible to decompose a sphere into finitely many pieces and reassemble the pieces to form two spheres identical to the first. In fact 5 pieces are enough, but they cannot be Lebesgue measurable. Could there be then such a decomposition where the pieces are projective, that is definable in analysis? (See Wagon [23].)

Discussion: to summarise, using some obvious abbreviations, we have a list of 5 Problems.

P1: $LM(PROJ)$
P2: $BP(PROJ)$
P3: $PSP(PROJ)$
P4: $Unif(PROJ)$
P5: *Banach Tarski with projective pieces*.

Each of these questions deals with subject matter that is familiar to mathematical analysts in the 21st century, and has been so since the early 20th. One of my points in introducing these is to make clear that such questions are themselves clear. The Continuum Problem is usually wheeled out to serve as a stalking horse for the difficulties of a realist view of set theory, or at least of the real continuum, that an author wishes to introduce. However in logical terms the Continuum Problem is a problem in *third order* number theory: one must use an existential quantifier ranging over subsets of the real line. The problems above are expressible without requiring such quantifications to take place, they are expressible in *second order number theory* or commonly called *analysis*: the quantifiers range over sets of numbers, or over functions from $\mathbb{N}$ to $\mathbb{N}$ and the complexity, that is the number of quantifier alternations, is the number of the rank of the set in the Lusin projective hierarchy being discussed (roughly speaking). The Real Continuum is often spoken of (*cf.* Feferman [2]) as having potentially an "indeterminate nature" since its third order statement relies upon the supposed mysteries of the power set operation when applied to an infinite set. "What is the cardinality of $\mathcal{P}(\omega)$?" The problems above are however of a more concrete nature. Analysts rarely come across questions that turn upon the cardinality of the continuum. They come across questions about the Lebesgue measurability of definable sets, that is sets within $PROJ$, on a daily basis. And they commonly recognise analytic and co-analytic sets as being tractable, as they enjoy these regularity properties. Thus the Problems listed are concrete problems within, and stated within, mathematical analysis.

3 Difficulties

We shall see that notwithstanding the 'simpler' logical second order definition of the concepts involved in these problems, they are subject to the same independence phenomena as the third order Continuum Problem, and, as we shall see, for roughly the same reasons. (That is: on the one hand there is an *inner model* of the universe of sets in which the continuum hypothesis was true, namely Gödel's L - such inner models are transitive subclasses of V which are models of the ZFC axioms; and on the other hand there are techniques derived from Cohen's *forcing method* where he showed the consistency of the negation of CH with the other axioms. The same dichotomy appears below: on one hand appeal to the model L to get one answer, and forcing techniques the consistency of the other.)

Thus: the Regularity Properties can consistently fail:

Theorem 3.1 *(Gödel) If ZF is consistent, then so is ZFC+*"There is a projective set that is not LM".

Indeed there is a *projection of a co-analytic* ("PCA") set that fails to be LM. This gives a negative "answer" to P1. The reason being that in Gödel's universe of constructible sets, L, with which he showed the consistency of the axioms of ZF together with CH, there are non-Lebesgue measurable sets at roughly the level of the complexity of the wellordering of the universe of L that he also demonstrated existed. Recall that the construction of the Vitali non-measurable set uses a wellordering of the continuum. Thus one expects a failure of Lebesgue measurability at roughly the same level of complexity as the wellordering of the continuum we have in L, which is used to construct a Vitali counter-example.

This use of the wellorder of L also leads to the following propositions relating to the problems above. In L:
P2: there is a PCA-set without the Baire property BP;
P3: there is a co-analytic set that is uncountable with no perfect subset;
P5: there is a paradoxical decomposition of the unit sphere in $\mathbb{R}^3$ using PCA-pieces.

For P4 the matter is slightly more nuanced: For co-analytic sets in the plane or higher dimensions, we have seen by the Novikov-Kondo theorem that they are uniformisable by co-analytic functions. For higher levels the wellorder of L ensures that sets in the projective classes PCA, $PCPCA$... *etc.* are all uniformisable by functions in the same class. (And if those hold for a suitable class Γ it is a straightforward result that it must fail for their complements in co-Γ.) A more delicate question for the Uniformisation Problem is to ask that the uniformising function come from the *very same* class or level in the projective hierarchy as the set being uniformised. Then by Novikov-Kondo this holds for co-analytic sets; in L this also holds for PCA sets (and for further classes on the repeated projected side: $PCPCA$ ... *etc.* .)

Whereas Gödel's construction of L gives a canonical *inner model* of V - the universe of all sets of mathematical discourse, there are various constructions based on extensions of Cohen's *method of forcing* which allow one to conclude that *consistent with* the axioms of ZF is the possibility that various levels of the projective hierarchy can be all Lebesgue measurable, or enjoy the other regularity properties.

Indeed a renowned theorem of Solovay allows that *all* sets are Lebesgue measurable and have the Baire property:

Theorem 3.2 (Solovay (1964) [19],[21]) *If the theory ZF+* "There is an inaccessible cardinal" *is consistent, then so is the theory ZF + DC+* "Every set that is LM and has the BP".

The above is quite remarkable. The DC is 'Dependent Choice' that allows for an infinite *sequence* of choices in any given relation $R(v_1, v_0)$ to be made. This is usually - but not always - all that an analyst requires. (The full Axiom of Choice is paradoxically, usually only invoked to guarantee the existence of pathological sets, *i.e.* difficult sets that are not LM or do not have the regularity properties *etc.*) The extra assumption beyond ZFC of the inaccessible cardinal was queried for many years as to its necessity. Eventually Shelah [18] showed that it was needed for Lebesgue measurability of all sets but not for the Baire property (thus breaking what had seemed a tight link, that what was true for sets of one kind was true of the other.)

However, these are only consistency results, and do not tell us about the facts of the matter in V. Notwithstanding this, mathematicians might simply have shifted to a view that all sets that they could write down and specify were LM and BP and used DC with comfort. But they seemingly have not.

Solovay also showed that, even retaining the full axiom of choice all definable sets of reals have strong regularity properties:

Theorem 3.3 (Solovay (1964) [19],[21]) *If the theory $ZF+$* "There is an inaccessible cardinal" *is consistent, then so is the theory $ZFC+$* "Every set projective set is LM and has the BP".

4 Resolution and Reflection

4.1 Resolution

Are there principles that are somehow missing from the ZFC axioms, and that could resolve these problems? For many set theorists the assumption that all sets are in Gödel's L - which indeed resolves these problems in a somewhat negative direction - is unpalatable. The iterative conception of set of sets appearing ever upwards in increasing ranks in a hierarchy built along the ordinals using the power set operation:

$$V_0 = \varnothing; \quad V_{\alpha+1} = P(V_\alpha); \quad Lim(\lambda) \to V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$

has an entirely mathematical feel to it. The construction of L replaces the successor step with only allowing sets definable in first order logic rather than the full power set. However why should this purely logical construction deliver all the sets that there are, sets which arise from a purely mathematical set theoretical conception?

It is a well known part of this story that Gödel himself allowed for the possibility that *strong axioms* might settle questions such as CH. However then a discussion then ensures about the justification of these strong axioms. At such a length of time since Zermelo's formulations of the axioms for sets, and with the additions of Skolem, and Fraenkel, it seems inconceivable that any basic fact of sets has been overlooked in the ZFC system. Any supplementing axioms may have to have a different set of justifications or grounds for acceptance. It is usual at this point to talk about *intrinsic* grounds that follow from the iterative conception of set and the V hierarchy as outlined as above, or more widely 'set-structure' concerning the whole of $(V, \in)$. These are to be contrasted with *extrinsic* grounds where the consequences of these hypotheses are so rich and so compelling that we feel we should to adopt them.

There is much to be said (and has been) at this point but we shall pass over this. Our targeted aim is that certain viewpoints of the universe $(V, \in)$ encourage a view that 'large cardinals' or 'strong axioms of infinity' can be invoked by 'reflecting' on the universe. Solovay delivered a striking clue in an early theorem relying on the assumption of a measurable cardinal:

Theorem 4.1 (Solovay [20]) *ZF proves that if there is a $<$-κ-additive 2-valued measure on some set of cardinality $\kappa > \aleph_0$ then $BP(PCA)$, $LM(PCA)$, $PSP(PCA)$.*

These conclusions are then quite in contradiction to the picture given in Gödel's L. There is also a mystery as to why the existence of measures, or equivalently ultrafilters on fields of sets quite remote from $V_{\omega+1}$ (which contains all the real numbers, or elements of Baire space or ...) should affect properties down at this very modest rank.

As oblique as the idea at first appears the determinacy of *two person perfect information games* implies much about the regularity properties of the real continuum.

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ (or some $X^{\mathbb{N}}$). The game G_A is defined as follows:

$$\begin{array}{lllll} I \text{ plays } k_0 & k_2 & \dots k_{2n} \dots & & \\ II \text{ plays } & k_1 & k_3 & \dots k_{2n+1} \dots \end{array}$$

- *I wins* if and only if $x = (k_0, k_1, \dots) \in A$.
- G_A is *determined* if either Player has a winning strategy in this game.

Let us write, for a class of set Γ "$Det(\Gamma)$" for the statement that for every set $A \in \Gamma$ that the game G_A is determined. "$Det(PROJ)$" is then read as "Projective Determinacy" or sometimes "Definable Determinacy".

Theorem 4.2 (Mycielski [13], [14]) *$Det(PROJ)$ implies Regularity for the projective sets.*

Thus the Solovay theorem from a measurable cardinal, and the results from assuming Definable Determinacy were leading in the same direction. The following indicated that these matters were no coincidence;

Theorem 4.3 (Martin [9]) *ZF proves that if there is a $<$-κ-additive 2-valued measure on some set of cardinality $\kappa > \aleph_0$ then $Det(Analytic)$.*

This was much earlier than the landmark theorem of Martin:

Theorem 4.4 (Martin [10]) *ZF proves $Det(Borel)$.*

(This remains the most quotable theorem in mathematics that *requires ZF* - as H. Friedman had previously shown ([3]) that ω_1-many iterations of the power set operation together with appropriate instances of Replacement would be required.)

However ZFC is just not strong enough to prove $Det(Analytic)$ on its own: this is because $Det(Analytic)$ can prove the consistency of ZFC. (And we cannot contradict Gödel's Incompleteness Theorems.) After much effort the prize was won:

Theorem 4.5 (Martin-Steel [11]) *If there are infinitely many Woodin cardinals then $Det(PROJ)$ and hence Regularity for the projective sets.*

Theorem 4.6 (Woodin [27]) *If there are infinitely many Woodin cardinals and a measurable cardinal above them, then in $L(\mathbb{R})$, the Gödel closure of $\mathbb{R}$ through all the ordinals, G_A is determined for* every $A \subseteq \mathbb{R}$. *And hence Regularity for all sets in $L(\mathbb{R})$.*

Thus "*AD*", the axiom that games based on any sets are determined, and which thus implies the regularity properties for all sets, is consistent with DC (as it holds in $L(\mathbb{R})$) but not the full AC. We could note also, that as strategies for such games can themselves be construed as sets of integers, or reals, that AD holding in $L(\mathbb{R})$, is equivalent to the statement that all games that are definable in $L(\mathbb{R})$ are determined (in V). We thus may prove outright the regularity properties from sufficient large cardinals. But we may be thought to have replaced a collection of problems by problems yet more problematic: how to justify the existence of such cardinals in the universe $(V, \in)$?

4.2 Reflection

To say that the universe of all sets is an unfinished totality does not mean objective undeterminateness, but merely a subjective inability to finish it.
Gödel, in (Wang: "A Logical Journey: From Gödel to Philosophy").

We take the view that the ordinals for example, indeed form a determinate concept: they are the class of sets that are transitive and wellordered by set membership. They form a proper class as Cantor and Burali-Forti (the latter eventually) recognised. We denote by On the totality of all ordinals.

Historically *reflection principles* are associated with attempts to formulate the idea that no one notion, idea, statement can capture our whole view of $V = \bigcup_{\alpha \in On} V_\alpha$. Such reflection principles are usually formulated in some language (first or higher order) as positing that sentences φ (when interpreted in the appropriate way over V) that hold in $\langle V, \in, \ldots \rangle$, must also hold in some $\langle V_\beta, \in, \ldots \rangle$. Let us call this *sentential* reflection. This is again a broad subject, and the reader is directed to Koellner's article 'Reflection Principles' ([5]) for a more in-depth discussion of the possible scope and limitations of reflection principles. Koellner argues that principles that may be deemed of an 'intrinsic nature' are unable to deliver any strong axioms that are inconsistent with a view that $V = L$, and so are not strong enough by themselves to prove outright anything about the real continuum beyond what we can already *in L*.

We first review some of the traditional sentential reflection principles.

(1) Montague-Levy: First order Reflection.
(R_0) : *For any* $\varphi(v_0, \ldots, v_n) \in \mathcal{L}_{\dot\in}$

$$ZF \vdash \forall \alpha \exists \beta > \alpha \forall \vec{x} \in V_\beta [\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}].$$

First order Reflection is actually provable in ZF. Indeed if we drop Infinity and the Replacement Scheme from ZF, the resulting theory, when augmented by (R_0), gives back Infinity and Replacement. It is a scheme-theorem and thus a metatheorem: it is a theorem only with one φ at a time. However by formalising a Σ_n-Satisfaction predicate we have:

For each n, $\mathrm{ZF} \vdash \exists C_n [C_n \subseteq On$ *is a c.u.b. class so that for any* $\varphi \in \mathrm{Fml}_{\Sigma_n}$:

$$\forall \beta \in C_n \forall \vec{x} \in V_\beta [\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}]].$$

Informally we write this as $\forall \beta \in C_n : (V_\beta, \in) \prec_{\Sigma_n} (V, \in)$.

(2) Levy, Bernays Reflection.

Suppose we allow some *second order* methods and allow *proper classes* to enter the picture more actively. If we allow reflection on classes then we can deliver some modest large cardinals. Let $\Phi(D)$ be the assertion that:

"D is a function from On to On, but $\forall \alpha$ D"α is bounded in On".

By the Axiom of Replacement for any class D, we have: Then

$$(V, \in, D) \models \Phi(D).$$

If we allow the assumption that $\forall D \Phi(D)$ *reflects* to some V_κ we shall have:

$$\forall D \subseteq V_\kappa \, (V_\kappa, \in, D) \models \Phi(D).$$

This implies that κ is a *strongly inaccessible cardinal*. However strongly inaccessible cardinals are strongly inaccessible in L, and are thus consistent with "$V = L$". The strict ZF-ist will eschew such an argument as it quantifies over all classes and not all such are necessarily definable over $(V, \in)$.

Whilst Levy [6] remained at the level of discussing Reflection to obtain inaccessible cardinals, and inductively defined hierarchies of such principles relating to the much earlier cardinals of Paul Mahlo, Bernays ([1]) allowed Φ above to be any Π^1_n formula about some parameter D. The resulting strengthened reflection principle now goes by the name of an *indescribability* property: any Π^1_n-property may be reflected downwards. Indeed there are Π^1_{n+1} sentences, that if reflected over $(V, \in)$ to some $(V_\kappa, \in)$ ensure that $(V_\kappa, \in)$ itself is Π^1_n-indescribable in the same sense.[1] The point to note here is that we have firmly entered the realm of second order entities: we must use these to realise the second order variables of our language, and moreover we must have a *domain of quantification* for the string of quantifiers in such a sentence to vary over. It is quite possible to consider third, fourth, n'th order languages over $(V, \in)$ and the associated reflection principles. But then such a layering of ranks of classes above V leads to the inevitable question as to why we do not declare such layers to be inhabited by sets.

We shall see that it is part of our viewpoint to avoid even second order methods wherever possible. We swallow the logical necessity of the existence of classes, as Cantor, Russell, and Burali-Forti showed, and admit of two types of objects: the mathematical realm of *sets* which constitute the universe of mathematical discourse $(V, \in)$; but we consider classes as just the *parts* of V (in a mereological fashion), which themselves may or may not be sets.

Gödel again:

> *All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory.* The other principles are only heuristic principles. *Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases.* Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now."
>
> (Section 8.7.9 of Wang [24]).

(Our italics.)

> The Universe of sets cannot be uniquely characterized (i.e. distinguished from all of its initial segments) by any internal structural property of the membership relation in it,

[1] We have not exactly delineated modern indescribability properties here, which usually are defined with an extra free predicate symbol, but we only wish to give the flavour of this. See Kanamori [4], I.6, for a fuller discussion.

> which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.
>
> (Wang - *ibid.*)

> Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is indefinable, where definability is to be taken in [a] more and more generalized and idealized sense. (Wang, *ibid.*, p. 285)

Gödel is presumed to be happy with considering logics of higher types, and thus Bernays may not be overstepping that mark. But the reflection we have proposed is not one of a logical character, meaning in a logic of higher types, or in an infinitary language; it is a structural reflection that takes the above 'unknowability' of the first quote above, or we prefer: 'ineffability', of the whole universe of sets, together with its parts, and reflects on that structure to bring it down to a set sized substructure.

Strengthening Reflection Principles

As we alluded to above, Koellner ([5]) has outlined a heuristic argument that intrinsic justifications of reflection will never produce a justification for a large cardinal that cannot reside in L, The cardinals or principles produced will all be consistent with $V = L$ (and he argues that the small large cardinals they could conceivably justify are technically weaker than an ω-Erdős cardinal).

The Challenge then: To justify a set-theoretic *reflection principle* that will ensure the existence of large cardinals (or strong axioms of infinity) that are sufficient to deliver the hypotheses needed for modern set theoretical principles.

We first mention some recent attempts at strengthening reflection. Notwithstanding Bernays' higher order reflection of sentences, Reinhardt pointed out that for formulae with third order parameters, the reflection scheme was inconsistent ([16]). Tait ([22]) attempted to provide some relief from this by placing restrictions on the substitutions possible and defined syntactic classes of higher order formulae with parameters on which one could nevertheless reflect, and showed the consistency of some of these principles from a measurable cardinal, and left the consistency of others open. Koellner ([5]) showed the latter inconsistent but proved the consistency of the former from a so-called ω-Erdős cardinal, which we do not define here, but is a cardinal again that is consistent with $V = L$. He further gave a heuristic argument that any reflection principle that was based on the intrinsic iterative hierarchy of sets, and which include such sentential reflections, are all limited in their outcomes, would have to be intraconstructible as its consistency would be derivable from such an ω-Erdős cardinal. Such cardinals then would never be not strong enough to prove outright anything about the real continuum beyond what we can *in L*.

To summarise we thus have:

- The Reflection Principles to date are all consistent with a view of the universe as being L the constructible one: they are *intra-constructible.*
- However these are all motivated on a syntactic level.

The moral is thus: We need stronger Reflection Principles: those that generalise Montague-Levy are not up to the task of providing any justification for the large cardinals needed for modern set theory.

We have proposed ([26], [25]) a *Global Reflection Principle* to overcome this intraconstructibility limitation. This principle has as its inspiration, the properties related to those of a subcompact cardinal, and is more of the nature of *structural reflection* rather than sentential or linguistic.

It is quite legitimate to ask first why do we need stronger reflection principles? Here is one very good reason.

Theorem 4.7 (Woodin) *Suppose there is a proper class of Woodin cardinals. Then* $Th(L(\mathbb{R}))$ *is immune to change by set forcing.*

The import of the theorem is that the perhaps baleful effects of Cohen's forcing method can have no effect on the fact of the matter as to which sentences are true in analysis, or indeed of any statement about the reals and ordinals: in $L(\mathbb{R})$ every object is definable from such, and as such it encompasses analysis, the projective hierarchy, and way beyond, using iterated definability through On. This is thus a strong *absoluteness* result. Whilst Woodin cardinals have a somewhat tricky definition (which is why we have not defined them here) it turns out that this notion is absolutely central to proving the consistency of many concepts in modern set theory. The supposition of their existence, or indeed that they are unbounded in the ordinals, is now ubiquitous in current theorems of set theory.

We shall therefore intend to define such a *Global Reflection Principle* (GRP) which will deliver an unbounded class of such large cardinals. We take an almost naive Cantorian stance, and consider the *absolute infinities* that he identified at that time: the absolute infinity of On the ordinals, $Card$ the class of cardinals, V itself *etc.* , and we collect these (without as yet being too precise as to what this means), into a family $\mathcal{C}$, but our viewpoint will be that $\mathcal{C}$ is the collection of the mereological parts of V. Some of these will be set-sized and we simplify matters by simply identifying them with the corresponding sets. Those parts that are not sized are the interesting entities in $\mathcal{C}$, and we think of these as the proper classes. We then consider a *structural reflection* of the whole universe $(V, \in, \mathcal{C})$ together with its parts to a small structure.

Global Reflection Principle - GRP

We take a small (meaning *set-sized*) substructure of $(V, \in, \mathcal{C})$, the universe with all of its parts, $\mathcal{C}$, and ask that this is then isomorphic to a small part of V: namely some V_α together with all of its parts. The 'parts' of V_α are naturally those $D \subseteq V_\alpha$, that is $V_{\alpha+1}$. The language $\mathcal{L}$ in which we wish to state the principle's reflection properties is the usual first order language for set theory, but augmented with predicate variables $X_0, X_1, X_2, \ldots$ that will vary over the collection $\mathcal{C}$. It is important to note that there are no second order quantifers over these variables. We thus avoid having an explicitly demarked domain of quantification for the second order objects. We thus write Σ^0_ω for the class of formulae of $\mathcal{L}$. In the second line of the next definition the first structure is thus 'Σ^0_ω'-elementary in the second, meaning as usual that such formulae with substitutions for set variables from X and for predicate variables from $\mathcal{C}'$ have the same truth value in both structures.

Definition 4.8 (Global Reflection Principle - GRP) *There is a set $X \subseteq V$ and a collection $\mathcal{C}' \subseteq \mathcal{C}$ with :*

$$(X, \in, \mathcal{C}') \prec_{\Sigma^0_\omega} (V, \in, \mathcal{C})$$

and:

$$(X, \in) = (V_\alpha, \in)$$

for some $\alpha \in On$, and so that

$$V_{\alpha+1} = \{D \cap V_\alpha \mid D \in \mathcal{C}'\}.$$

This can be summarised as we have a transitivising isomorphism π so that

$$\pi : (X, \in, \mathcal{C}') \longrightarrow (V_\alpha, \in, V_{\alpha+1})$$

with π the identity on X.

Hence we have

$$(V, \in, \mathcal{C}) \text{ is reflected down to } (V_\alpha, \in, V_{\alpha+1})$$

Or to put it another way, we are thus requiring that there is set-sized simulacrum of

$$(V, \in, \mathcal{C}) \text{ that is of the form } (V_\alpha, \in, V_{\alpha+1}).$$

Indeed the inverse of π yields an *elementary embedding* in an equivalent formulation that is perhaps more congenial to set theorists:

There is an initial segment of the universe V_α, and a nontrivial elementary embedding

$$\pi^{-1} : (V_\alpha, \in, V_{\alpha+1}) \longrightarrow_{\Sigma^0_\omega} (V, \in, \mathcal{C})$$

with critical point α (i.e., $\pi^{-1}(\alpha) > \alpha$ whereas for $z \in V_\alpha$, π^{-1} is the identity: $\pi^{-1}(z) = z$).

Thus all that π^{-1} does is move, or stretch, objects from $V_{\alpha+1}$ to objects in $\mathcal{C}$. Equivalently, as there are no 'points' above V_α in X, π's collapsing action on any $X \in \mathcal{C}$ satisfies $\pi(X) = X \cap V_\alpha$. Thus, for example the part of V which is the class of ordinals, On, is in $\mathcal{C}$, and $\pi^{-1}(\alpha) = On$. (Where α here is considered a class over V_α and as an element of $V_{\alpha+1}$.) We thus have

$$\varphi(\vec{x}, \vec{X})^{(V_\alpha, \in, V_{\alpha+1})} \leftrightarrow \varphi(\pi^{-1}(\vec{x}), \pi^{-1}(\vec{X}))^{(V, \in, \mathcal{C})} \leftrightarrow \varphi(\vec{x}, \pi^{-1}(\vec{X}))^{(V, \in, \mathcal{C})}.$$

Why GRP? Define a field of classes U on $\mathcal{P}(\alpha)$ by

$$X \in U \leftrightarrow \alpha \in \pi^{-1}(X).$$

As $\mathcal{P}(\alpha) \subseteq V_{\alpha+1} \subseteq \mathrm{dom}(\pi^{-1})$ by Σ^0_1-elementarity (in π^{-1}), this is an ultrafilter. Standard arguments show that U is a normal measure on α, and thus α is a measurable cardinal. But then:

$\forall \beta < \alpha \;\langle V, \in \rangle \vDash$"$\exists \alpha > \beta (\alpha$ a measurable cardinal)" $\Rightarrow$
$\Rightarrow \langle V_\alpha, \in \rangle \vDash$" $\forall \beta \exists \lambda > \beta (\lambda$ a measurable cardinal)" $\Rightarrow$
$\Rightarrow \langle V, \in \rangle \vDash$"*There are unboundedly many measurable cardinals*".

It is an exercise in the appropriate definitions to show that the critical point α is also a Woodin (indeed a Shelah) cardinal. So we thus have:

Theorem 4.9 (GRP) $(V, \in) \models \forall\alpha\exists\lambda > \alpha(\lambda$ *a measurable Woodin cardinal).*

Corollary 4.10 *By the results of Martin-Steel and Woodin mentioned above, GRP then implies:*

- *a) Projective Determinacy* $Det(PROJ)$ *and* $(AD)^{L(\mathbb{R})}$.
- *b) (Woodin)* $\mathrm{Th}(L(\mathbb{R}))$ *is fixed: no set forcing notion can change* $\mathrm{Th}(L(\mathbb{R}))$, *and in particular the truth value of any sentence about reals in the language of analysis, thereby including* $Det(PROJ)$.

5 Discussion

There is a discussion to be had as to whether we have here a genuine reflection principle. As we have intimated, the nature of the reflection that is occurring is logical in as much as it relies on passing from the whole universe V (with its parts) to a substructure that preserves a certain amount of logical elementarity, namely Σ^0_ω-elementarity. But it is structural in that it requires the substructure to be isomorphic to an initial segment of the universe V_α together with *all* of V_α's classes, that is $V_{\alpha+1}$. It is quite possible to posit weaker reflection principles where the second order domain of π^{-1} is only a proper subset of $V_{\alpha+1}$. It might for example only include $P(\alpha)^L$ for example. Whereas this would be enough to deduce the existence of $0^\sharp$, that is the existence of a non-trivial embedding $\pi^{-1} : L \longrightarrow L$, we could not define the measure on $P(\alpha)$ as we did above.

We could view GRP as the natural limit of a series of principles where we demanded more and more classes of V_α to be in the image of π (whilst perhaps allowing α to vary to achieve this). Thus larger and larger inner models M would have M-measures defined on their $P(\alpha)^M$ as we just saw for L yielding $0^\sharp$ above.

We did not quantify over the collection $\mathcal{C}$ in any fashion. All we required of it was that it contained sufficient elements to allow the definition of the GRP. This allowed us to be somewhat vague as to what the collection $\mathcal{C}$ was. The status of $\mathcal{C}$ is discussed in [25] and [26]. There we discuss various approaches as to how to regard $\mathcal{C}$, for example through the manœuvre of considering plurals and plural quantification. However we reject this in favour of a mereological approach. This fulfils the need to find a way to sufficiently distinguish sets from classes (see [8] for a discussion on this). The argument often deployed against considering higher order types over the top of V is that in such a case 'there was no reason to stop building V at the level On' (or some such). By thinking of the ordinals as a determinate concept, we have a class of sets priorly given in a mathematical manner. In this viewpoint there are no 'ordinals beyond V': V is the universe of all mathematical objects, and ordinals are mathematical objects. Likewise there is a mathematical power set operation $P(x)$, but a 'power-class operation' acting on the parts of V would be something else altogether different, and would not be considered a mathematical operation.

We did not (yet) formalise GRP in any class theory. We think of the development of our intuitions concerning V and its parts as taking place in a pre-formalised state: these include are our intuitions concerning the ineffability of $(V, \in, \mathcal{C})$, and are not yet formalised. We think simply as mathematicians do about the semantics, or structure of our concepts.

One could then proceed to formalise the statement of GRP in NBG - class theory. The assertion of the existence of such a π appears *prima facie* to be third order, but with usual coding tricks, in fact it can be regarded as an assertion that a certain kind of class exists and thus is a Σ^1_1 statement. It is not hard to come up with strengthenings: we could for example increase the amount of elementarity demanded to include that of a language with full quantification over the variables X_i. This 'mereological reflection' then has reflection involving quantified statements about the parts of V. This is natural, but a stronger principle than GRP. However then one must be specific about the domain of quantification, *i.e.* the class $\mathcal{C}$. Our own inclination is to eschew second order methods whenever possible - being not entirely convinced of their coherence. One point that can be made is that with $(V_\alpha, \in, V_{\alpha+1})$ being a natural model of Morse-Kelley class theory, if we have an enhanced version of GRP with this form of full second order reflection, we can carry this up to deduce that $(V, \in, \mathcal{C})$ must also form a model of Morse-Kelly.

A final technical word on the consistency of GRP. One can easily see that:

Theorem 5.1

$$Con(ZFC + \exists\kappa(\kappa \textit{ is 1-extendible})) \longrightarrow Con(NBG + GRP).$$

Indeed 1-extendibility is stronger than the enhanced versions of GRP mentioned above. Sam Roberts in [17] has extended the above account of global reflection to yield a very flexible family of higher order reflection principles which, when suitably formulated, can yield supercompact cardinals and more. But these cross the philosophical threshold we have stopped short of: not to quantify over the parts of V. They also cross over the admittedly more technical set theoretical threshold into those cardinals, such as supercompacts, that imply the existence of embeddings j of the universe into an inner model that are discontinuous at the successor cardinal of the critical point (the first ordinal moved) of j. Such embeddings require representation by systems of ultrapowers known as 'long extenders', which we do not define here, but for a discussion of 'long' and 'short' extender types see Section 1 of [15]. The GRP here falls just short of justifying such. The GRP embedding is of a 'superstrong' type where On is the target of the critical point κ and is at the limit of those that can be expressed using short extenders. Here a cardinal κ is called *superstrong* if there is an embedding j, preserving elementarity in the usual first order language of set theory, with critical point κ, and an inner model M so that $j : V \longrightarrow M$ with $V^M_{j(\kappa)} = V_{j(\kappa)}$. Note now that $j \restriction V_{\kappa+1} : V_{\kappa+1} \longrightarrow V^M_{j(\kappa)+1}$ however without any assumption that the latter is $V_{j(\kappa)+1}$. (The latter would require 1-extendibility.) But taking $\mathcal{C} = V^M_{j(\kappa)+1}$, it is easy to see $(V_{j(\kappa)}, \in, \mathcal{C})$ together with $j \restriction V_{\kappa+1}$ gives a set model of $NBG + GRP$.

Hence the last theorem can be modestly improved, in that the antecedent is a large cardinal that can be expressed by short extenders.

Theorem 5.2

$$Con(ZFC + \exists\kappa(\kappa \textit{ is superstrong})) \longrightarrow Con(NBG + GRP).$$

It is possible given such a j from GRP to define from it a directed system of short extenders from V, in such a way that this directed system expresses an ultrapower embedding which when restricted to $V_{\kappa+1}$ is just j. We thus get back j from this directed system (see the discussion in [4] Ch. 5 Sect. 26 for example.) Were long extender embeddings ever to be shown, as a class, inconsistent, then GRP is pretty much what we would be left with.

References

[1] P. Bernays. Zur Frage der Unendlichkeitsschemata in der axiomatische Mengenlehre. In *Essays on the Foundations of Mathematics*, pages 3–49. Magnus Press, Hebrew University of Jerusalem, 1961. 10

[2] S. Feferman. Is the Continuum Hypothesis a definite mathematical problem? *EFI Workshop papers, Harvard*, 2012. 5

[3] H. Friedman. Higher set theory and mathematical practice. *Annals of Mathematical Logic*, 2(3):325–327, 1970. 8

[4] A. Kanamori. *The Higher Infinite*. Springer Monographs in Mathematics. Springer Verlag, New York, 2nd edition, 2003. 10, 15

[5] P. Koellner. On reflection principles. *Ann. Pure Appl. Logic*, 157:206–219, 2009. 9, 11

[6] A. Levy. Axiom schemata of strong infinity in axiomatic set theory. *Pacific Jouranl of Mathematics*, 10:223–238, 1960. 10

[7] N. Lusin and W. Sierpiński. Sur un ensemble non measurable B. *Journal de Mathématiques, 9e serie*, 2:53–72, 1923. 3

[8] P. Maddy. Proper classes. *Journal for Symbolic Logic*, 48(1):113–139, March 1983. 14

[9] D.A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970. 8

[10] D.A. Martin. Borel determinacy. *Annals of Mathematics*, 102:363–371, 1975. 8

[11] D.A. Martin and J.R. Steel. A proof of Projective Determinacy. *Journal of the American Mathematical Society*, 2:71–125, 1989. 8

[12] Y.N. Moschovakis. *Descriptive Set Theory*. Studies in Logic series. North-Holland, Amsterdam, 2009. 2

[13] J. Mycielski. On the axiom of determinateness. *Fundamenta Mathematicae*, 53:205–224, 1964. 8

[14] J. Mycielski. On the axiom of determinateness II. *Fundamenta Mathematicae*, 59:203–212, 1966. 8

[15] I. Neeman. Determinacy in $L(\mathbb{R})$. In M. Magidor M. Foreman, A. Kanamori, editor, *Handbook of Set Theory*, volume III, chapter 22. Springer Verlag, Berlin, New York, 2007. 15

[16] W. Reinhardt. Remarks on reflection principles, large cardinals, and elementary embeddings. In T. Jech, editor, *Axiomatic Set Theory*, volume 13 part 2 of *Proceedings of Symposia in Pure Mathematics*, pages 189–205, Providence, Rhode Island, 1974. American Mathematical Society. 11

[17] S. Roberts. A strong reflection principle. *Review of Symbolic Logic*, 10(4):651–662, 2017. 15

[18] S. Shelah. Can you take Solovay's inaccessible away? *Israel Journal of Mathematics*, 48:1–47, 1984. 6

[19] R.M. Solovay. The measure problem (abstract). *Notices of the American Mathematical Society*, 12:217, 1965. 6, 7

[20] R.M. Solovay. The cardinality of Σ^1_2 sets of reals. In J.J. Bulloff, T.C. Holyoke, and S.W. Hahn, editors, *Foundations of Mathematics: symposium papers commemorating the sixtieth birthday of Kurt Gödel*, pages 58–73. Springer-Verlag, 1969. 7

[21] R.M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92:1–56, 1970. 6, 7

[22] W.W. Tait. Constructing cardinals from below. In W. W. Tait, editor, *The Provenance of Pure Reason: essays in the philosophy of mathematics and its history*, pages 133–154. Oxford University Press, Oxford, 2005. 11

[23] S. Wagon. *The Banach-Tarski Paradox*. Cambridge University Press, 1994. 5

[24] H. Wang. *A Logical Journey: From Godel to Philosophy*. MIT Press, Boston, USA, 1996. 10

[25] P.D. Welch. Global Reflection Principles. *Isaac Newton Institute Pre-print Series.*, Exploring the Frontiers of Incompleteness, EFI Workshop Papers, Harvard,(INI12051-SAS), 2012. 12, 14

[26] P.D. Welch and L. Horsten. Reflecting on Absolute Infinity. *Journal of Philosophy*, 113:89–111, February 2016. 12, 14

[27] W.H. Woodin. Supercompact cardinals, sets of reals, and weakly homogeneous trees. *Proceedings of the National Academy of Sciences of the United States of America*, 85(18):6587–6591, 1988. 8

[28] W.H. Woodin. The Continuum Hypothesis, Part I. *Notices of the American Mathematical Society*, pages 567–576, July 2001. 2

[29] W.H. Woodin. The Continuum Hypothesis, Part II. *Notices of the American Mathematical Society*, pages 681–690, August 2001. 2